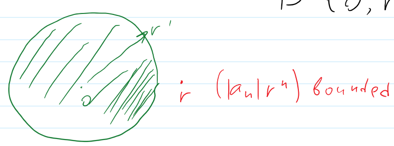
Power series

Saturday, September 9, 2023 9:04 AM

Def. Let $(a_n)_{n\geq 0}$ be a sequence of complex numbers.
$\sum_{n=0}^{\infty} a_n z^n$ is a (formal) power series.

$S_n(z) = \sum_{k=0}^{n} a_k z^k$ – nth partial sum, a polynomial.

Lemma. Let for some $r \geq 0$, $\{|a_n| r^n\}$ is bounded.
Then for any z with $|z| < r$, $\sum a_n z^n$ converges.
Moreover, if $r' < r$, then the series converges uniformly on $\overline{B}(0, r')$.

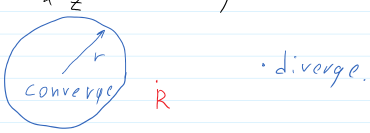


Remark. Not on $\overline{B}(0,r)$ or even $B(0,r)$! Consider $\sum z^n$, $r = 1$

Proof. When $|z| \leq r'$, $|a_n z^n| \leq |a_n (r')^n| = |a_n r^n| \left(\frac{r'}{r}\right)^n < C\left(\frac{r'}{r}\right)^n$
$\sum C\left(\frac{r'}{r}\right)^n < \infty$, so we can use M-test ∎

Note: $\{|a_n| r^n\}$ – not bounded $\Rightarrow$ for z with $|z| = r$,
$\sum a_n z^n$ diverge ($|a_n z^n| = |a_n| r^n \not\to 0$).

Theorem. For any power series $\sum a_n z^n$ let
$R := \sup\{r : \text{the sequence } (|a_n| r^n) \text{ is bounded}\}$. Then
1) $\sum a_n z^n$ converges uniformly on $\overline{B}(0,r)$ for any $r < R$.
2) $\sum a_n z^n$ diverges for any $|z| > R$.



Def R is called radius of convergence.

Proof. 1) $r < R \Rightarrow \exists\, r < r' < R \Rightarrow r' \in \{t : (|a_n| t^n) \text{ is bounded}\} \Rightarrow$
$(|a_n| (r')^n)$ – bounded $\underset{\text{Lemma}}{\Longrightarrow}$ 1)

2) $|z| > R \Rightarrow |z| \notin \{t : (|a_n| t^n) \text{ is bounded}\} \Rightarrow (|a_n| |z|^n)$ – unbounded $\Rightarrow$ 2)

Cauchy-Hadamard formula for radius of convergence.

$R = \dfrac{1}{\overline{\lim} \sqrt[n]{|a_n|}}$ (Convention for this formula: $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$)

Jacques Salomon Hadamard

Proof.

$$r < \frac{1}{\overline{\lim}\sqrt[n]{|a_n|}} \Rightarrow \overline{\lim}\, r\sqrt[n]{|a_n|} < 1 \Rightarrow \exists a<1, N: n>N \Rightarrow r^n|a_n| < a^n < 1.$$

$$\overline{\lim}\, r\sqrt[n]{|a_n|} = \overline{\lim}\sqrt[n]{r^n|a_n|}$$

So $(r^n|a_n|)$ is bounded.

So $r \le R \Rightarrow R \ge \frac{1}{\overline{\lim}\sqrt[n]{|a_n|}}$

If $(r^n|a_n|)$ - bounded, $r^n|a_n| \le M \Rightarrow r\sqrt[n]{|a_n|} \le M^{\frac{1}{n}} \Rightarrow$

$\overline{\lim}\, r\sqrt[n]{|a_n|} \le \lim_{n\to\infty} M^{\frac{1}{n}} = 1$. So $r \le \frac{1}{\overline{\lim}\sqrt[n]{|a_n|}}$.

So $\frac{1}{\overline{\lim}\sqrt[n]{|a_n|}}$ is an upper bound for $\{r : (|a_n|r^n) \text{ bounded}\}$ ■

$\sum a_n z^n$ has radius of convergence $R = \frac{1}{\overline{\lim}_{n\to\infty}\sqrt[n]{|a_n|}}$.

converges uniformly in any disk $\{|z| \le r\}$, $r < R$.

continuous in $\{|z| < R\}$.

$|z| < R$.

Lemma Let $|z| \le r$, $|w| \le r$. Then

$|z^n - w^n| \le n|z-w|r^{n-1}$.

Proof $(z^n - w^n) = (z-w)(z^{n-1} + z^{n-2}w + \dots + w^{n-1})$

So $|z^n - w^n| \le |z-w| \sum_{k=0}^{n-1} \underbrace{|w^k z^{n-k-1}|}_{\le r^{n-1}} \le n r^{n-1}|z-w|$ ■

Theorem. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with the radius of convergence R. Then $f(z)$ is analytic on $B(0,R)$ and $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ (termwise derivative).

Proof. Let $f_1(z) := \sum_{n=1}^{\infty} n a_n z^{n-1}$.

First, observe that the radius of convergence of $\sum_{n=1}^{\infty} n a_n z^{n-1}$ is R.

Proof. Let R_1 be the radius of convergence of $\sum_{n=1}^{\infty} n a_n z^{n-1}$. Then, since $|na_n| \ge |a_n|$, $R_1 \le R$.

On the other hand, if $r < R$, then take $r_0 : r < r_0 < R$

$$\lim_{n\to\infty} n|a_n|r^n = \lim_{n\to\infty} n|a_n|r_0^n \cdot \frac{r^n}{r_0^n} = \lim_{n\to\infty} |a_n|r_0^n \cdot \lim_{n\to\infty} n\left(\frac{r}{r_0}\right)^n = 0 \cdot 0$$

So $r < R_1$. So we have: $r < R \Rightarrow r < R_1 \Rightarrow R_1 \ge R \Rightarrow R_1 = R$.

$\lim n q^n = 0$ if $q < 1$.

Proof $q = \frac{1}{1+\delta}$, $\delta > 0$

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$\lim n q^n = 0$ if $q < 1$.

Proof $q = \frac{1}{1+\delta}$, $\delta > 0$

$(1+\delta)^n > \frac{n(n-1)}{2}\delta^2$.

So $n q^n < \frac{2n}{n(n-1)\delta^2} = \frac{2}{\delta^2(n-1)} \to 0$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{k} a_n z^n + \sum_{n=k+1}^{\infty} a_n z^n = S_k(z) + R_k(z).$$

$f_1(z) := \sum_{n=1}^{\infty} n a_n z^{n-1}$ $\quad S_n'(z)$ - nth partial sum for f_1.

Let $|z|, |z_0| < r < R$

$$\frac{f(z)-f(z_0)}{z-z_0} - f_1(z_0) = \left(\frac{S_n(z)-S_n(z_0)}{z-z_0} - S_n'(z_0)\right) + \quad \mathrm{I}$$

$$\left(S_n'(z_0) - f_1(z_0)\right) + \quad \mathrm{II}$$

$$\frac{R_n(z)-R_n(z_0)}{z-z_0} \quad \mathrm{III}$$

$$\left|\frac{R_n(z)-R_n(z_0)}{z-z_0}\right| = \left|\sum_{k=n+1}^{\infty} \frac{a_k(z^k - z_0^k)}{z-z_0}\right| \overset{\text{Lemma}}{\leq} \sum_{k=n+1}^{\infty} |a_k| k r^{k-1}$$

Now: fix $\varepsilon > 0$:

Find n: $\quad |S_n'(z_0) - f_1(z_0)| < \frac{\varepsilon}{3}$ (Partial sum!)

$\sum_{k=n+1}^{\infty} k|a_k| r^{k-1} < \frac{\varepsilon}{3}$ $\quad (r < R = R_1)$

Find $\delta > 0$: $\quad |z - z_0| < \delta \Rightarrow \left|\frac{S_n(z)-S_n(z_0)}{z-z_0} - S_n'(z_0)\right| < \frac{\varepsilon}{3}$

[using differentiability of S_n.

Thus $|z - z_0| < \delta \Rightarrow$

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - f_1(z_0)\right| \leq |\mathrm{I}| + |\mathrm{II}| + |\mathrm{III}| < \varepsilon.$$

So $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f_1(z_0)$

Corollary (Taylor) $\sum a_n z^n$ is infinitely differentiable for $|z| < R$,

$$f^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n z^{n-k}$$

$$a_n = \frac{f^{(n)}(0)}{n!}$$

Brook Taylor

Proof. Induction. Plug in $z = 0$. $\blacksquare$